

# A WEAK GALERKIN FINITE ELEMENT METHOD FOR A TYPE OF FOURTH ORDER PROBLEM ARISING FROM FLUORESCENCE TOMOGRAPHY

CHUNMEI WANG\* AND HAOMIN ZHOU†

**Abstract.** In this paper, a new and efficient numerical algorithm by using weak Galerkin (WG) finite element methods is proposed for a type of fourth order problem arising from fluorescence tomography (FT). Fluorescence tomography is an emerging, in vivo non-invasive 3-D imaging technique which reconstructs images that characterize the distribution of molecules that are tagged by fluorophores. Weak second order elliptic operator and its discrete version are introduced for a class of discontinuous functions defined on a finite element partition of the domain consisting of general polygons or polyhedra. An error estimate of optimal order is derived in an  $H^2$ -equivalent norm for the WG finite element solutions. Error estimates in the usual  $L^2$  norm are established, yielding optimal order of convergence for all the WG finite element algorithms except the one corresponding to the lowest order (i.e., piecewise quadratic elements). Some numerical experiments are presented to illustrate the efficiency and accuracy of the numerical scheme.

**Key words.** weak Galerkin, finite element methods, fourth order problem, weak second order elliptic operator, fluorescence tomography, polygonal or polyhedral meshes.

**AMS subject classifications.** Primary, 65N30, 65N15, 65N12, 74N20; Secondary, 35B45, 35J50, 35J35

**1. Introduction.** This paper is concerned with new developments of numerical methods for a type of fourth order problem with Dirichlet and Neumann boundary conditions. The model problem seeks an unknown function  $u = u(x)$  satisfying

$$(1.1) \quad \begin{aligned} (-\nabla \cdot (\kappa \nabla) + \mu)^2 u &= f, & \text{in } \Omega, \\ u &= \xi, & \text{on } \partial\Omega, \\ \kappa \nabla u \cdot \mathbf{n} &= \nu, & \text{on } \partial\Omega, \end{aligned}$$

where  $\Omega$  is an open bounded domain in  $\mathbb{R}^d$  ( $d = 2, 3$ ) with a Lipschitz continuous boundary  $\partial\Omega$ ,  $\mathbf{n}$  is the unit outward normal direction to  $\partial\Omega$ ,  $\kappa$  is a symmetric and positive definite matrix-valued function,  $\mu$  is nonnegative real-valued function, and the functions  $f$ ,  $\xi$ , and  $\nu$  are given in the domain or on its boundary, as appropriate. For convenience, denote the second order elliptic operator  $\nabla \cdot (\kappa \nabla)$  as  $E$ . For simplicity and without loss of generality, we assume that  $\kappa$  is piecewise constant matrix and  $\mu$  is a piecewise constant.

The fourth order model problem (1.1) arises from fluorescence tomography (FT) [5, 10, 8, 13, 17, 28, 29, 31], which is an emerging, in vivo noninvasive 3-D imaging technique. FT captures molecular specific information by using highly specific flu-

\*School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia, 30332, USA; Taizhou College, Nanjing Normal University, Taizhou, China 225300. The research of Chunmei Wang was supported by National Science Foundation Award DMS-1522586 and by Jiangsu Provincial Foundation Award BK20050538.

†School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA. The research of Haomin Zhou was supported by NSF Faculty Early Career Development (CAREER) Award DMS-0645266, DMS-1042998, DMS-1419027, and ONR Award N000141310408.

orescent probes and non-ionizing NIR radiation instead of X-ray or other powerful magnetic field [12], which makes FT a potentially less harmful medical imaging modality compared to other medical imaging modalities, such as CT and MRI. FT aims to reconstruct the distribution of fluorophores, which are tagged with targeted molecules, from boundary measurements. Therefore, FT has been regarded as a promising method in early cancer detection and drug monitoring nowadays [2, 11, 30].

We introduce the following space

$$H_\kappa^2(\Omega) = \{v : v \in H^1(\Omega), \kappa \nabla v \in H(\text{div}; \Omega)\},$$

which is equipped with the following norm

$$\|v\|_{\kappa,2} = (\|v\|_1^2 + \|\nabla \cdot (\kappa \nabla v)\|^2)^{\frac{1}{2}}.$$

A variational formulation for the fourth order model problem (1.1) is given by seeking  $u \in H_\kappa^2(\Omega)$  satisfying  $u|_{\partial\Omega} = \xi$ ,  $\kappa \nabla u \cdot \mathbf{n}|_{\partial\Omega} = \nu$ , such that

$$(1.2) \quad (Eu, Ev) + 2\mu(\kappa \nabla u, \nabla v) + \mu^2(u, v) = (f, v), \quad \forall v \in \mathcal{V},$$

where  $(\cdot, \cdot)$  stands for the usual inner product in  $L^2(\Omega)$ , and the test space  $\mathcal{V}$  can be defined as

$$\mathcal{V} = \{v \in H_\kappa^2(\Omega) : v|_{\partial\Omega} = 0, \kappa \nabla v \cdot \mathbf{n}|_{\partial\Omega} = 0\}.$$

Here,  $\mathbf{n}$  is the unit outward normal direction to the boundary of  $T$ .

There have been various conforming finite element schemes proposed for a general fourth order elliptic problem, such as the biharmonic equation, by constructing finite element spaces as subspaces of  $H^2(\Omega)$ . Such  $H^2$ -conforming methods essentially require  $C^1$ -continuity for the underlying piecewise polynomials (known as finite element functions) on a prescribed finite element partition [3]. The  $C^1$ -continuity imposes an enormous difficulty in the construction of the corresponding finite element functions in practical computations due to the fact that the  $C^1$ -continuous elements have high degrees of freedom. For example, the Argyris element has 21 degrees of freedom, and the Bell element has 18 degrees of freedom. Because of the complexity in the construction of  $C^1$ -continuous elements,  $H^2$ -conforming finite element methods are rarely used in practice for solving the biharmonic equation. As an alternative approach, nonconforming and discontinuous Galerkin finite element methods have been developed for solving the biharmonic equation over the last several decades. The Morley element [19] is a well-known example of nonconforming element for the biharmonic equation by using piecewise quadratic polynomials. Recently, a  $C^0$  interior penalty method was studied in [4, 6]. In [20], a hp-version interior-penalty discontinuous Galerkin method was developed for the biharmonic equation. To avoid the use of  $C^1$ -elements, mixed methods have been developed for the biharmonic equation by reducing the fourth order problem to a system of two second order equations [1, 7, 9, 18, 14].

The difference between (1.2) and the standard bi-harmonic equation is significant. First, the usual  $H^2$  conforming elements designed for the bi-harmonic equation are no longer  $H_\kappa^2$ -conforming, and thus are not applicable to the problem (1.2). For some well-known non-conforming finite elements, such as the Morley element [19], for the biharmonic equation, the corresponding variational formulation involves the full Hessian. Since it is not clear if the problem (1.2) can be re-formulated in an equivalent

form, the applicability of such non-conforming finite elements is highly questionable. In fact, we believe that they can not be directly applicable to the problem (1.2). The problem (1.2) can also be formulated in a mixed form by using an auxiliary variable  $w = -\nabla \cdot (\kappa \nabla u) + \mu u$ . The exact mixed formulation seeks  $u, w \in H^1(\Omega)$  such that  $u|_{\partial\Omega} = \xi$  and satisfying

$$(1.3) \quad \begin{aligned} (w, \phi) - (\kappa \nabla u, \nabla \phi) - (\mu u, \phi) &= -\langle \nu, \phi \rangle_{\partial\Omega}, & \forall \phi \in H^1(\Omega), \\ (\kappa \nabla w, \nabla v) + (\mu w, v) &= (f, v), & \forall v \in H_0^1(\Omega). \end{aligned}$$

Most of the existing finite element methods are applicable to the mixed formulation (1.3). One drawback with the mixed formulation is the saddle-point nature of the problem, which causes extra difficulty in the design of fast solution techniques for the corresponding discretizations.

Recently, weak Galerkin (WG) finite element method is a new and efficient numerical method for solving partial differential equations. The method/idea was first proposed by Junping Wang and Xiu Ye in 2011 for solving second order elliptic problem in [25], and the concept was further developed in [16, 21, 22, 23, 24, 26, 27]. The central idea of WG is to interpret partial differential operators as generalized distributions, called weak differential operators, over the space of discontinuous functions including boundary information. The weak differential operators are further discretized and applied to the corresponding variational formulations of the underlying PDEs. By design, WG uses generalized and/or discontinuous approximating functions on general meshes to overcome the barrier in the construction of “smooth” finite element functions. The current research indicates that the concept of discrete weak differential operators offers a new paradigm in numerical methods for partial differential equations.

The proposed WG finite element algorithm for the fourth order problem (1.1) is based on two new ideas: (1) the computation of a discrete weak second order elliptic operator locally on each element that takes into account the coefficient matrix from applications; and (2) a stabilizer that takes into account the jump of the coefficient matrix from applications. The research is innovative in that the proposed algorithm is the first ever finite element method that works for the fourth order problem (1.1) based on the variational formulation (1.2).

There is a point which is of great significance to point out in the process of our research. Actually, there is a more straightforward variational formulation for the fourth order problem (1.1) which is easier to propose: seeking  $u \in H_\kappa^2(\Omega)$  satisfying  $u|_{\partial\Omega} = \xi$ ,  $\kappa \nabla u \cdot \mathbf{n}|_{\partial\Omega} = \nu$ , such that

$$(1.4) \quad (Fu, Fv) = (f, v), \quad \forall v \in \mathcal{V},$$

where  $F = -\nabla \cdot (\kappa \nabla) + \mu$ . However, according to our research, we find that variational formulation (1.4) is not an appropriate variational formulation where weak Galerkin finite element method can be applied successfully and the code based on variational formulation (1.4) cannot work well especially when  $\mu$  is not small enough. There is a straightforward reason from the formulation that there is no second order elliptic term in the variational formulation (1.4) which plays an important role in the algorithm design. This fact leads to the result that the analysis of WG method cannot pass through based on the variational formulation (1.4).

The paper is organized as follows. Section 2 is devoted to a discussion of weak second order elliptic operator, and weak gradient as well as their discrete versions. In Section 3, we present a weak Galerkin algorithm for the fourth order model problem (1.1) based on the variational formulation (1.2). In Section 4, we introduce some local  $L^2$  projection operators and then derive some approximation properties which are useful in the convergence analysis. Section 5 will be devoted to the derivation of an error equation for the WG finite element solution. In Section 6, we shall establish an optimal order of error estimate for the WG finite element approximation in a  $H^2$ -equivalent discrete norm. In Section 7, we shall derive error estimates for the WG finite element solution in the usual  $L^2$ -norm. Finally in Section 8, we present some numerical results that verify the theory established in the previous sections.

Throughout the paper, the letter  $C$  is used to denote a generic constant independent of the mesh size and functions involved.

**2. Weak Differential Operators and Discrete Weak Differential Operators.** For the fourth order model problem (1.1) with the variational form (1.2), the principle differential operators are the second order elliptic operator  $E$ , and the gradient operator. Thus, we shall define weak second order elliptic operator and review the definition of weak gradient operator, which was first introduced in [26], for a class of discontinuous functions. For numerical purpose, we shall also introduce a discrete version for the weak second order elliptic operator and review the definition of discrete weak gradient operator [26] in the polynomial subspaces.

Let  $\mathcal{T}_h$  be a partition of the domain  $\Omega$  into polygons in 2D or polyhedra in 3D. Assume that  $\mathcal{T}_h$  is shape regular in the sense as defined in [26], which is more complex than the standard definition of shape regularity in the theory of finite element methods. Denote by  $\mathcal{E}_h$  the set of all edges or flat faces in  $\mathcal{T}_h$ , and let  $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$  be the set of all interior edges or flat faces in  $\mathcal{T}_h$ . By a weak function on the region  $T$ , we mean a function  $v = \{v_0, v_b, v_g\}$  such that  $v_0 \in L^2(T)$ ,  $v_b \in L^2(\partial T)$  and  $v_g \in L^2(\partial T)$ . The first and second components  $v_0$  and  $v_b$  can be understood as the value of  $v$  in the interior and on the boundary of  $T$ . The third term  $v_g$  intends to represent the value of  $\kappa \nabla v \cdot \mathbf{n}$  on the boundary of  $T$ , where  $\mathbf{n}$  is the unit outward normal direction to the boundary of  $T$ . On each interior edge or flat face  $e \in \mathcal{E}_h^0$  shared by two elements denoted by  $T_L$  and  $T_R$ ,  $v_g$  has two copies of value: one as seen from the left-hand side element  $T_L$  denoted by  $v_g^L$ , and the other as seen from the right-hand side element  $T_R$  denoted by  $v_g^R$ , such that  $v_g^L + v_g^R = 0$ .

Denote by  $W(T)$  the space of all weak functions on  $T$ ; i.e.,

$$W(T) = \{v = \{v_0, v_b, v_g\} : v_0 \in L^2(T), v_b \in L^2(\partial T), v_g \in L^2(\partial T)\}.$$

**DEFINITION 2.1.** *The dual of  $L^2(T)$  can be identified with itself by using the standard  $L^2$  inner product as the action of linear functionals. With a similar interpretation, for any  $v \in W(T)$ , the weak second order elliptic operator of  $v = \{v_0, v_b, v_g\}$ , denoted by  $E_w v$ , is defined as a linear functional in the dual space of  $H^2(T)$  whose action on each  $\varphi \in H^2(T)$  is given by*

$$(2.1) \quad (E_w v, \varphi)_T = (v_0, E\varphi)_T - \langle v_b, \kappa \nabla \varphi \cdot \mathbf{n} \rangle_{\partial T} + \langle v_g, \varphi \rangle_{\partial T}.$$

Here,  $\langle \cdot, \cdot \rangle_{\partial T}$  stands for the usual inner product in  $L^2(\partial T)$ .

For computational purpose, we introduce a discrete version of the weak second order elliptic operator by approximating  $E_w$  in a polynomial subspace of the dual of  $H^2(T)$ . To this end, for any non-negative integer  $r \geq 0$ , denote by  $P_r(T)$  the set of polynomials on  $T$  with degree no more than  $r$ . A discrete weak second order elliptic operator, denoted by  $E_{w,r,T}$ , is defined as the unique polynomial  $E_{w,r,T}v \in P_r(T)$  satisfying the following equation

$$(2.2) \quad (E_w v, \varphi)_T = (v_0, E\varphi)_T - \langle v_b, \kappa \nabla \varphi \cdot \mathbf{n} \rangle_{\partial T} + \langle v_g, \varphi \rangle_{\partial T}, \quad \forall \varphi \in P_r(T),$$

which follows from the usual integration by parts that

$$(2.3) \quad (E_w v, \varphi)_T = (E v_0, \varphi)_T + \langle v_0 - v_b, \kappa \nabla \varphi \cdot \mathbf{n} \rangle_{\partial T} - \langle \kappa \nabla v_0 \cdot \mathbf{n} - v_g, \varphi \rangle_{\partial T}, \quad \forall \varphi \in P_r(T).$$

**DEFINITION 2.2.** [26] *The dual of  $L^2(T)$  can be identified with itself by using the standard  $L^2$  inner product as the action of linear functionals. With a similar interpretation, the weak gradient operator of any  $v \in W(T)$ , denoted by  $\nabla_w v$ , is defined as a linear vector functional in the dual space of  $[H^1(T)]^d$  whose action on each  $\psi \in [H^1(T)]^d$  is given by*

$$(2.4) \quad (\nabla_w v, \psi)_T = -(v_0, \nabla \cdot \psi)_T + \langle v_b, \psi \cdot \mathbf{n} \rangle_{\partial T}.$$

A discrete weak gradient operator, denoted by  $\nabla_{w,r,T}$ , is defined as the unique vector polynomial  $\nabla_{w,r,T}v \in [P_r(T)]^d$  satisfying the following equation

$$(2.5) \quad (\nabla_w v, \psi)_T = -(v_0, \nabla \cdot \psi)_T + \langle v_b, \psi \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \psi \in [P_r(T)]^d,$$

which follows from the usual integration by parts that

$$(2.6) \quad (\nabla_w v, \psi)_T = (\nabla v_0, \psi)_T - \langle v_0 - v_b, \psi \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \psi \in [P_r(T)]^d.$$

**3. Numerical Algorithm by Weak Galerkin.** For any given integer  $k \geq 2$ , denote by  $W_k(T)$  the local discrete weak function space given by

$$W_k(T) = \{v = \{v_0, v_b, v_g\} : v_0 \in P_k(T), v_b \in P_k(e), v_g \in P_{k-1}(e), e \subset \partial T\}.$$

Patching  $W_k(T)$  over all the elements  $T \in \mathcal{T}_h$  through the interface  $\mathcal{E}_h^0$ , we arrive at a weak finite element space  $V_h$  defined as follows

$$V_h = \{v = \{v_0, v_b, v_g\} : \{v_0, v_b, v_g\}|_T \in W_k(T), \forall T \in \mathcal{T}_h\}.$$

Denote by  $V_h^0$  the subspace of  $V_h$  with vanishing trace; i.e.,

$$V_h^0 = \{v = \{v_0, v_b, v_g\} \in V_h, v_b|_e = 0, v_g|_e = 0, e \subset \partial T \cap \partial\Omega\}.$$

Denote by  $E_{w,k-2}$  and  $\nabla_{w,k-1}$  the discrete weak second order elliptic operator and the discrete weak gradient operator on the finite element space  $V_h$  computed by using (2.2) and (2.5) on each element  $T$  for  $k \geq 2$ , respectively; i.e.,

$$(E_{w,k-2}v)|_T = E_{w,k-2,T}(v|_T), \quad v \in V_h,$$

$$(\nabla_{w,k-1}v)|_T = \nabla_{w,k-1,T}(v|_T), \quad v \in V_h.$$

For simplicity of notation and without confusion, we shall drop the subscripts  $k-2$  and  $k-1$  in the notations  $E_{w,k-2}$  and  $\nabla_{w,k-1}$ , respectively. For any  $u = \{u_0, u_b, u_g\}$  and  $v = \{v_0, v_b, v_g\}$  in  $V_h$ , we introduce the following notations and a bilinear form as follows

$$(E_w u, E_w v)_h = \sum_{T \in \mathcal{T}_h} (E_w u, E_w v)_T,$$

$$(\kappa \nabla_w u, \nabla_w v)_h = \sum_{T \in \mathcal{T}_h} (\kappa \nabla_w u, \nabla_w v)_T,$$

$$s(u, v) = \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle \kappa \nabla u_0 \cdot \mathbf{n} - u_g, \kappa \nabla v_0 \cdot \mathbf{n} - v_g \rangle_{\partial T} + \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle u_0 - u_b, v_0 - v_b \rangle_{\partial T}.$$

For each element  $T$ , denote by  $Q_0$  the  $L^2$  projection onto  $P_k(T)$ . For each edge or face  $e \subset \partial T$ , denote by  $Q_b$  and  $Q_g$  the  $L^2$  projections onto  $P_k(e)$  and  $P_{k-1}(e)$ , respectively. Now, for any  $u \in H^2(\Omega)$ , we can define a projection onto the weak finite element space  $V_h$  such that on each element  $T$ ,

$$Q_h u = \{Q_0 u, Q_b u, Q_g(\kappa \nabla u \cdot \mathbf{n})\}.$$

The following is a precise statement of the WG finite element scheme for the fourth order model problem (1.1) based on the variational formulation (1.2).

**WEAK GALERKIN ALGORITHM 1.** Find  $u_h = \{u_0, u_b, u_g\} \in V_h$  satisfying  $u_b = Q_b \xi$  and  $u_g = Q_g \nu$  on  $\partial\Omega$ , such that

$$(3.1) \quad (E_w u_h, E_w v)_h + 2\mu(\kappa \nabla_w u_h, \nabla_w v)_h + \mu^2(u_h, v) + s(u_h, v) = (f, v_0), \forall v \in V_h^0.$$

The following is a useful observation concerning the finite element space  $V_h^0$ .

**LEMMA 3.1.** For any  $v \in V_h^0$ , define  $\|v\|$  by

$$(3.2) \quad \|v\|^2 = (E_w v, E_w v)_h + 2\mu(\kappa \nabla_w v, \nabla_w v)_h + \mu^2(v, v) + s(v, v).$$

Then,  $\|\cdot\|$  defines a norm in the linear space  $V_h^0$ .

*Proof.* It is easily seen that  $\|v\|$  defines a semi norm on the finite element space  $V_h^0$  when we write the term  $2\mu(\kappa \nabla_w v, \nabla_w v)_h$  as  $2\mu(\kappa^{\frac{1}{2}} \nabla_w v, \kappa^{\frac{1}{2}} \nabla_w v)_h$ . We shall only verify the positivity property for  $\|\cdot\|$ . To this end, assume that  $\|v\| = 0$  for some  $v \in V_h^0$ . It follows from (3.2) that  $E_w v = 0$  on each element  $T$ ,  $\kappa \nabla v_0 \cdot \mathbf{n} = v_g$  and  $v_0 = v_b$  on each  $\partial T$ . Thus, for any  $\varphi \in P_{k-2}(T)$ , from (2.3), we obtain

$$\begin{aligned} 0 &= (E_w v, \varphi)_T \\ &= (E v_0, \varphi)_T + \langle v_0 - v_b, \kappa \nabla \varphi \cdot \mathbf{n} \rangle_{\partial T} - \langle \kappa \nabla v_0 \cdot \mathbf{n} - v_g, \varphi \rangle_{\partial T} \\ &= (E v_0, \varphi)_T, \end{aligned}$$

which implies that  $Ev_0 = 0$  on each element  $T$ . Thus, from integration by parts, we obtain

$$\begin{aligned}
0 &= \sum_{T \in \mathcal{T}_h} (Ev_0, v_0)_T \\
&= \sum_{T \in \mathcal{T}_h} -(\kappa \nabla v_0, \nabla v_0)_T + \langle \kappa \nabla v_0 \cdot \mathbf{n}, v_0 \rangle_{\partial T} \\
&= \sum_{T \in \mathcal{T}_h} -(\kappa \nabla v_0, \nabla v_0)_T + \langle \kappa \nabla v_0 \cdot \mathbf{n} - v_g, v_0 \rangle_{\partial T} \\
&= \sum_{T \in \mathcal{T}_h} -(\kappa \nabla v_0, \nabla v_0)_T,
\end{aligned}$$

where we have used the fact that the sum for the terms associated with  $v_g$  vanishes (note that  $v_g$  vanishes on  $\partial T \cap \partial \Omega$ ). This implies that  $\nabla v_0 = 0$  on each element  $T$ . Thus,  $v_0$  is a constant on each element  $T$ , which, together with the fact that  $v_0 = v_b$  on each  $\partial T$ , indicates that  $v_0$  is continuous over the whole domain  $\Omega$ . Thus, we obtain that  $v_0 = C$  in  $\Omega$ . Together with the facts that  $v_0 = v_b$  on  $\partial T$  and  $v_b|_{\partial T \cap \partial \Omega} = 0$ , we obtain  $v_0 = 0$  in  $\Omega$ , which, combining with the facts that  $v_0 = v_b$  on each  $\partial T$  and  $v_b|_{\partial T \cap \partial \Omega} = 0$  indicates that  $v_b = 0$  in  $\Omega$ . Furthermore, the facts that  $\kappa \nabla v_0 \cdot \mathbf{n} = v_g$  on each  $\partial T$  and  $v_g|_{\partial T \cap \partial \Omega} = 0$  yield  $v_g = 0$  in  $\Omega$ . Thus,  $v = 0$  in  $\Omega$ . This completes the proof of the lemma.  $\square$

LEMMA 3.2. *The weak Galerkin algorithm (3.1) has a unique solution.*

*Proof.* Let  $u_h^{(1)}$  and  $u_h^{(2)}$  be two different solutions of the weak Galerkin algorithm (3.1). It is clear that the difference  $e_h = u_h^{(1)} - u_h^{(2)}$  is a finite element function in  $V_h^0$  satisfying

$$(3.3) \quad (E_w e_h, E_w v)_h + 2\mu(\kappa \nabla_w e_h, \nabla_w v)_h + \mu^2(e_h, v) + s(e_h, v) = 0, \quad \forall v \in V_h^0.$$

By setting  $v = e_h$  in (3.3), we obtain

$$(E_w e_h, E_w e_h)_h + 2\mu(\kappa \nabla_w e_h, \nabla_w e_h)_h + \mu^2(e_h, e_h) + s(e_h, e_h) = 0.$$

From Lemma 3.1, we get  $e_h \equiv 0$ , i.e.,  $u_h^{(1)} = u_h^{(2)}$ . This completes the proof.  $\square$

**4.  $L^2$  Projections and Their Properties.** The goal of this section is to establish some technical results for the  $L^2$  projections. These results are valuable in the error analysis for the WG finite element method.

For any  $T \in \mathcal{T}_h$ , let  $\varphi$  be a regular function in  $H^1(T)$ . The following trace inequality holds true [26]:

$$(4.1) \quad \|\varphi\|_e^2 \leq C(h_T^{-1} \|\varphi\|_T^2 + h_T \|\nabla \varphi\|_T^2).$$

If  $\varphi$  is a polynomial on the element  $T \in \mathcal{T}_h$ , then we have from the inverse inequality (see also [26]) that

$$(4.2) \quad \|\varphi\|_e^2 \leq C h_T^{-1} \|\varphi\|_T^2.$$

Here  $e$  is an edge or flat face on the boundary of  $T$ .

On each element  $T \in \mathcal{T}_h$ , define  $\mathcal{Q}_h$  the local  $L^2$  projection onto  $P_{k-2}(T)$  and  $\mathcal{Q}_1$  the local  $L^2$  projection onto  $P_{k-1}(T)$ .

LEMMA 4.1. *The  $L^2$  projections  $\mathcal{Q}_h$ ,  $\mathcal{Q}_1$  and  $\mathcal{Q}_h$  satisfy the following commutative properties:*

$$(4.3) \quad E_w(Q_h w) = \mathcal{Q}_h(Ew), \quad \forall w \in H_\kappa^2(T),$$

$$(4.4) \quad \nabla_w(Q_h w) = \mathcal{Q}_1(\nabla w), \quad \forall w \in H^1(T).$$

*Proof.* As to (4.3), for any  $\varphi \in P_{k-2}(T)$  and  $w \in H_\kappa^2(T)$ , from the definition (2.2) of  $E_w$  and the usual integration by parts, we have

$$\begin{aligned} (E_w(Q_h w), \varphi)_T &= (Q_0 w, E\varphi)_T - \langle Q_b w, \kappa \nabla \varphi \cdot \mathbf{n} \rangle_{\partial T} + \langle Q_g(\kappa \nabla w \cdot \mathbf{n}), \varphi \rangle_{\partial T} \\ &= (w, E\varphi)_T - \langle w, \kappa \nabla \varphi \cdot \mathbf{n} \rangle_{\partial T} + \langle \kappa \nabla w \cdot \mathbf{n}, \varphi \rangle_{\partial T} \\ &= (\varphi, Ew)_T \\ &= (\varphi, \mathcal{Q}_h(Ew))_T. \end{aligned}$$

As to (4.4), for any  $\psi \in [P_{k-1}(T)]^d$  and  $w \in H^1(T)$ , from the definition (2.5) of  $\nabla_w$  and the usual integration by parts, we have

$$\begin{aligned} (\nabla_w(Q_h w), \psi)_T &= -(Q_0 w, \nabla \cdot \psi)_T + \langle Q_b w, \psi \cdot \mathbf{n} \rangle_{\partial T} \\ &= -(w, \nabla \cdot \psi)_T + \langle w, \psi \cdot \mathbf{n} \rangle_{\partial T} \\ &= (\psi, \nabla w)_T \\ &= (\psi, \mathcal{Q}_1(\nabla w))_T, \end{aligned}$$

which completes the proof.  $\square$

The following lemma provides some approximation properties for the projection operators  $\mathcal{Q}_h$ ,  $\mathcal{Q}_1$  and  $\mathcal{Q}_h$ .

LEMMA 4.2. [15, 26] *Let  $\mathcal{T}_h$  be a finite element partition of  $\Omega$  satisfying the shape regularity assumption as defined in [26]. Then, for any  $0 \leq s \leq 2$  and  $2 \leq m \leq k$ , there exists a constant  $C$  such that the following estimates hold true:*

$$(4.5) \quad \sum_{T \in \mathcal{T}_h} h_T^{2s} \|u - Q_0 u\|_{s,T}^2 \leq C h^{2(m+1)} \|u\|_{m+1}^2,$$

$$(4.6) \quad \sum_{T \in \mathcal{T}_h} h_T^{2s} \|Eu - \mathcal{Q}_h Eu\|_{s,T}^2 \leq C h^{2(m-1)} \|u\|_{m+1}^2,$$

$$(4.7) \quad \sum_{T \in \mathcal{T}_h} h_T^{2s} \|\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)\|_{s,T}^2 \leq C h^{2m} \|u\|_{m+1}^2.$$

LEMMA 4.3. *Let  $2 \leq m \leq k$  and  $u \in H^{\max\{m+1, 4\}}(\Omega)$ . There exists a constant  $C$  such that the following estimates hold true:*

$$(4.8) \quad \left( \sum_{T \in \mathcal{T}_h} h_T \|Eu - \mathcal{Q}_h(Eu)\|_{\partial T}^2 \right)^{\frac{1}{2}} \leq C h^{m-1} \|u\|_{m+1},$$



$$(4.9) \quad \left( \sum_{T \in \mathcal{T}_h} h_T^3 \|\kappa \nabla(Eu - \mathcal{Q}_h(Eu)) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \leq Ch^{m-1} (\|u\|_{m+1} + h\delta_{m,2}\|u\|_4),$$

$$(4.10) \quad \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\kappa \nabla(Q_0 u) \cdot \mathbf{n} - Q_g(\kappa \nabla u \cdot \mathbf{n})\|_{\partial T}^2 \right)^{\frac{1}{2}} \leq Ch^{m-1} \|u\|_{m+1},$$

$$(4.11) \quad \left( \sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_0 u - Q_b u\|_{\partial T}^2 \right)^{\frac{1}{2}} \leq Ch^{m-1} \|u\|_{m+1},$$

$$(4.12) \quad \left( \sum_{T \in \mathcal{T}_h} h_T^3 \|(\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \leq Ch^{m+1} \|u\|_{m+1}.$$

Here  $\delta_{i,j}$  is the usual Kronecker's delta with value 1 when  $i = j$  and value 0 otherwise.

*Proof.* To prove (4.8), by the trace inequality (4.1) and the estimate (4.6), we get

$$\begin{aligned} & \sum_{T \in \mathcal{T}_h} h_T \|Eu - \mathcal{Q}_h(Eu)\|_{\partial T}^2 \\ & \leq C \sum_{T \in \mathcal{T}_h} \|Eu - \mathcal{Q}_h(Eu)\|_T^2 + h_T^2 |Eu - \mathcal{Q}_h(Eu)|_{1,T}^2 \\ & \leq Ch^{2m-2} \|u\|_{m+1}^2. \end{aligned}$$

As to (4.9), by the trace inequality (4.1) and the estimate (4.6), we obtain

$$\begin{aligned} & \sum_{T \in \mathcal{T}_h} h_T^3 \|\kappa \nabla(Eu - \mathcal{Q}_h(Eu)) \cdot \mathbf{n}\|_{\partial T}^2 \\ & \leq C \sum_{T \in \mathcal{T}_h} h_T^2 \|\nabla(Eu - \mathcal{Q}_h(Eu))\|_T^2 + h_T^4 |\nabla(Eu - \mathcal{Q}_h(Eu))|_{1,T}^2 \\ & \leq Ch^{2m-2} (\|u\|_{m+1}^2 + h^2 \delta_{m,2} \|u\|_4^2). \end{aligned}$$

As to (4.10), by the trace inequality (4.1) and the estimate (4.5), we have

$$\begin{aligned} & \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\kappa \nabla(Q_0 u) \cdot \mathbf{n} - Q_g(\kappa \nabla u \cdot \mathbf{n})\|_{\partial T}^2 \\ & \leq \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\kappa \nabla(Q_0 u) \cdot \mathbf{n} - \kappa \nabla u \cdot \mathbf{n}\|_{\partial T}^2 \\ & \leq C \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\nabla(Q_0 u) - \nabla u\|_{\partial T}^2 \\ & \leq C \sum_{T \in \mathcal{T}_h} h_T^{-2} \|\nabla Q_0 u - \nabla u\|_T^2 + |\nabla Q_0 u - \nabla u|_{1,T}^2 \\ & \leq Ch^{2m-2} \|u\|_{m+1}^2. \end{aligned}$$

As to (4.11), by the trace inequality (4.1) and the estimate (4.5), we have

$$\begin{aligned}
& \sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_0 u - Q_b u\|_{\partial T}^2 \\
& \leq \sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_0 u - u\|_{\partial T}^2 \\
& \leq C \sum_{T \in \mathcal{T}_h} h_T^{-4} \|Q_0 u - u\|_T^2 + h_T^{-2} \|\nabla(Q_0 u - u)\|_T^2 \\
& \leq C h^{2m-2} \|u\|_{m+1}^2.
\end{aligned}$$

Finally, as to (4.12), by the trace inequality (4.1) and the estimate (4.7), we have

$$\begin{aligned}
& \sum_{T \in \mathcal{T}_h} h_T^3 \|(\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n}\|_{\partial T}^2 \\
& \leq \sum_{T \in \mathcal{T}_h} h_T^2 \|\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)\|_T^2 + h_T^4 \|\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)\|_{1,T}^2 \\
& \leq C h^{2m+2} \|u\|_{m+1}^2.
\end{aligned}$$

This completes the proof of the lemma.  $\square$

**5. An Error Equation.** Let  $u$  and  $u_h = \{u_0, u_b, u_g\} \in V_h$  be the solutions of (1.1) and (3.1), respectively. Denote by

$$(5.1) \quad e_h = Q_h u - u_h$$

the error function between the  $L^2$  projection of the exact solution  $u$  and its weak Galerkin finite element approximation  $u_h$ . An error equation refers to some identity that the error function  $e_h$  must satisfy. The goal of this section is to derive an error equation for  $e_h$ .

LEMMA 5.1. *The error function  $e_h$  as defined by (5.1) is a finite element function in  $V_h^0$  and satisfies the following equation*

$$(5.2) \quad (E_w e_h, E_w v)_h + 2\mu(\kappa \nabla_w e_h, \nabla_w v)_h + \mu^2(e_h, v)_h + s(e_h, v) = \phi_u(v), \quad \forall v \in V_h^0,$$

where

$$\begin{aligned}
(5.3) \quad \phi_u(v) = & - \sum_{T \in \mathcal{T}_h} \langle \kappa \nabla(Eu - \mathcal{Q}_h(Eu)) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T} \\
& + \sum_{T \in \mathcal{T}_h} \langle \kappa \nabla v_0 \cdot \mathbf{n} - v_g, Eu - \mathcal{Q}_h Eu \rangle_{\partial T} \\
& + \sum_{T \in \mathcal{T}_h} 2\mu \langle v_0 - v_b, (\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n} \rangle_{\partial T} + s(Q_h u, v).
\end{aligned}$$

*Proof.* Using (2.3) with  $\varphi = E_w(Q_h u)$ , from (4.3), we obtain

$$\begin{aligned}
(E_w v, E_w(Q_h u))_T = & (Ev_0, \mathcal{Q}_h(Eu))_T + \langle v_0 - v_b, \kappa \nabla(\mathcal{Q}_h(Eu)) \cdot \mathbf{n} \rangle_{\partial T} \\
& - \langle \kappa \nabla v_0 \cdot \mathbf{n} - v_g, \mathcal{Q}_h Eu \rangle_{\partial T} \\
= & (Ev_0, Eu)_T + \langle v_0 - v_b, \kappa \nabla(\mathcal{Q}_h(Eu)) \cdot \mathbf{n} \rangle_{\partial T} \\
& - \langle \kappa \nabla v_0 \cdot \mathbf{n} - v_g, \mathcal{Q}_h Eu \rangle_{\partial T},
\end{aligned}$$

which implies that

$$(5.4) \quad (Ev_0, Eu)_T = (E_w(Q_h u), E_w v)_T - \langle v_0 - v_b, \kappa \nabla(\mathcal{Q}_h(Eu)) \cdot \mathbf{n} \rangle_{\partial T} \\ + \langle \kappa \nabla v_0 \cdot \mathbf{n} - v_g, \mathcal{Q}_h Eu \rangle_{\partial T}.$$

Next, it follows from the integration by parts that

$$(5.5) \quad \sum_{T \in \mathcal{T}_h} (Eu, Ev_0)_T = \sum_{T \in \mathcal{T}_h} (E^2 u, v_0)_T - \langle \kappa \nabla(Eu) \cdot \mathbf{n}, v_0 \rangle_{\partial T} + \langle \kappa \nabla v_0 \cdot \mathbf{n}, Eu \rangle_{\partial T}.$$

Using (2.6) with  $\psi = \nabla_w(Q_h u)$ , from (4.4) and the integration by parts, we have

$$(5.6) \quad \sum_{T \in \mathcal{T}_h} 2\mu(\kappa \nabla_w(Q_h u), \nabla_w v)_T \\ = \sum_{T \in \mathcal{T}_h} 2\mu(\kappa \nabla v_0, \mathcal{Q}_1 \nabla u)_T - 2\mu\langle v_0 - v_b, \mathcal{Q}_1 \kappa \nabla u \cdot \mathbf{n} \rangle_{\partial T} \\ = \sum_{T \in \mathcal{T}_h} 2\mu(\nabla v_0, \kappa \nabla u)_T - 2\mu\langle v_0 - v_b, \mathcal{Q}_1 \kappa \nabla u \cdot \mathbf{n} \rangle_{\partial T} \\ = \sum_{T \in \mathcal{T}_h} -2\mu(v_0, \nabla \cdot (\kappa \nabla u))_T + 2\mu\langle v_0, \kappa \nabla u \cdot \mathbf{n} \rangle_{\partial T} - 2\mu\langle v_0 - v_b, \mathcal{Q}_1 \kappa \nabla u \cdot \mathbf{n} \rangle_{\partial T} \\ = \sum_{T \in \mathcal{T}_h} -2\mu(v_0, Eu)_T + 2\mu\langle v_0, (\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n} \rangle_{\partial T},$$

where we have used the fact that the sum for the terms associated with  $v_b$  vanishes (note that  $v_b$  vanishes on  $\partial T \cap \partial \Omega$ ).

From the definition of the projection, we get

$$(5.7) \quad \sum_{T \in \mathcal{T}_h} \mu^2(Q_h u, v)_T = \sum_{T \in \mathcal{T}_h} \mu^2(u_0, v_0)_T.$$

Adding (5.5)-(5.7) together and using the identity that

$$\sum_{T \in \mathcal{T}_h} (E^2 u, v_0)_T - 2\mu(Eu, v_0)_T + \mu^2(u_0, v_0)_T = (f, v_0),$$

we obtain

$$\sum_{T \in \mathcal{T}_h} (Eu, Ev_0)_T + 2\mu\kappa(\nabla_w(Q_h u), \nabla_w v)_T + \mu^2(Q_h u, v)_T \\ = (f, v_0) - \sum_{T \in \mathcal{T}_h} \left( \langle \kappa \nabla(Eu) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T} + \langle \kappa \nabla v_0 \cdot \mathbf{n} - v_g, Eu \rangle_{\partial T} \right. \\ \left. + 2\mu\langle v_0 - v_b, (\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n} \rangle_{\partial T} \right),$$

where we have used the fact that the sum for the terms associated with  $v_b$  and  $v_g$  vanishes (note that both  $v_b$  and  $v_g$  vanish on  $\partial T \cap \partial \Omega$ ). Combining the above equation

with (5.4) and adding  $s(Q_h u, v)$  to both sides of the equation yield

$$\begin{aligned}
& \sum_{T \in \mathcal{T}_h} (E_w(Q_h u), E_w v)_T + 2\mu\kappa(\nabla_w(Q_h u), \nabla_w v)_T + \mu^2(Q_h u, v)_T \\
& + s(Q_h u, v) \\
& = (f, v_0) - \sum_{T \in \mathcal{T}_h} \left( \langle \kappa \nabla(Eu - \mathcal{Q}_h(Eu)) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T} \right. \\
& \quad + \langle \kappa \nabla v_0 \cdot \mathbf{n} - v_g, Eu - \mathcal{Q}_h Eu \rangle_{\partial T} \\
& \quad \left. + 2\mu \langle v_0 - v_b, (\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n} \rangle_{\partial T} \right) + s(Q_h u, v),
\end{aligned}$$

which, subtracting (3.1), completes the proof.  $\square$

**6. Error Estimates in  $H^2$ .** The goal of this section is to derive some error estimates for the solution of weak Galerkin algorithm (3.1). The following result is an estimate for the error function  $e_h$  in the trip-bar norm which is essentially an  $H^2$ -equivalent norm in  $V_h^0$ .

**THEOREM 6.1.** *Let  $u_h \in V_h$  be the weak Galerkin finite element solution arising from (3.1) with finite elements of order  $k \geq 2$ . Assume that the exact solution  $u$  of (1.1) is sufficiently regular such that  $u \in H^{\max\{k+1, 4\}}(\Omega)$ . Then, there exists a constant  $C$  such that*

$$(6.1) \quad \|u_h - Q_h u\| \leq Ch^{k-1} \left( \|u\|_{k+1} + h\delta_{k,2} \|u\|_4 \right).$$

In other words, we have an optimal order of convergence in the  $H^2$ -equivalent norm.

*Proof.* Letting  $v = e_h$  in the error equation (5.2) gives rise to

$$\begin{aligned}
(6.2) \quad \|e_h\|^2 &= - \sum_{T \in \mathcal{T}_h} \langle \kappa \nabla(Eu - \mathcal{Q}_h(Eu)) \cdot \mathbf{n}, e_0 - e_b \rangle_{\partial T} \\
&+ \sum_{T \in \mathcal{T}_h} \langle \kappa \nabla e_0 \cdot \mathbf{n} - e_g, Eu - \mathcal{Q}_h Eu \rangle_{\partial T} \\
&+ \sum_{T \in \mathcal{T}_h} 2\mu \langle e_0 - e_b, (\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n} \rangle_{\partial T} \\
&+ \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle \kappa \nabla Q_0 u \cdot \mathbf{n} - Q_g(\kappa \nabla u \cdot \mathbf{n}), \kappa \nabla e_0 \cdot \mathbf{n} - e_g \rangle_{\partial T} \\
&+ \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle Q_0 u - Q_b u, e_0 - e_b \rangle_{\partial T}.
\end{aligned}$$

Each term on the right-hand side of (6.2) can be estimated as follows. For the first term of the right-hand side of (6.2), using the Cauchy-Schwarz inequality, and the estimate (4.9), one arrives at

$$\begin{aligned}
& \left| \sum_{T \in \mathcal{T}_h} \langle \kappa \nabla(Eu - \mathcal{Q}_h(Eu)) \cdot \mathbf{n}, e_0 - e_b \rangle_{\partial T} \right| \\
& \leq \left( \sum_{T \in \mathcal{T}_h} h_T^3 \|\kappa \nabla(Eu - \mathcal{Q}_h(Eu)) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^{-3} \|e_0 - e_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \leq Ch^{k-1} (\|u\|_{k+1} + h\delta_{k,2} \|u\|_4) \|e_h\|.
\end{aligned}$$

For the second term of the right-hand side of (6.2), using the Cauchy-Schwarz inequality, and the estimate (4.8), one arrives at

$$\begin{aligned} & \left| \sum_{T \in \mathcal{T}_h} \langle \kappa \nabla e_0 \cdot \mathbf{n} - e_g, Eu - \mathcal{Q}_h Eu \rangle_{\partial T} \right| \\ & \leq \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\kappa \nabla e_0 \cdot \mathbf{n} - e_g\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T \|Eu - \mathcal{Q}_h Eu\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ & \leq Ch^{k-1} \|u\|_{k+1} \|e_h\|. \end{aligned}$$

For the third term of the right-hand side of (6.2), using the Cauchy-Schwarz inequality, and the estimate (4.12), one arrives at

$$\begin{aligned} & \left| \sum_{T \in \mathcal{T}_h} 2\mu \langle e_0 - e_b, (\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n} \rangle_{\partial T} \right| \\ & \leq \left( \sum_{T \in \mathcal{T}_h} h_T^{-3} \|e_0 - e_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^3 \|(\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ & \leq Ch^{k+1} \|u\|_{k+1} \|e_h\|. \end{aligned}$$

For the fourth term of the right-hand side of (6.2), using the Cauchy-Schwarz inequality, and the estimate (4.10), one arrives at

$$\begin{aligned} & \left| \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle \kappa \nabla Q_0 u \cdot \mathbf{n} - Q_g(\kappa \nabla u \cdot \mathbf{n}), \kappa \nabla e_0 \cdot \mathbf{n} - e_g \rangle_{\partial T} \right| \\ & \leq Ch^{k-1} \|u\|_{k+1} \|e_h\|. \end{aligned}$$

For the last term of the right-hand side of (6.2), using the Cauchy-Schwarz inequality, and the estimate (4.11), one arrives at

$$\begin{aligned} & \left| \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle Q_0 u - Q_b u, e_0 - e_b \rangle_{\partial T} \right| \\ & \leq Ch^{k-1} \|u\|_{k+1} \|e_h\|. \end{aligned}$$

Substituting all the above estimates into (6.2) yields

$$\|e_h\|^2 \leq Ch^{k-1} (\|u\|_{k+1} + h\delta_{k,2}\|u\|_4) \|e_h\|,$$

which implies (6.1). This completes the proof of the theorem.  $\square$

**7. Error Estimates in  $L^2$ .** This section shall establish the estimates for the components  $e_0$ ,  $e_b$  and  $e_g$  of the error function  $e_h$  in the standard  $L^2$  norm, respectively. To this end, we consider the following dual problem:

$$(7.1) \quad \begin{aligned} F^2 \psi &= e_0, & \text{in } \Omega, \\ \psi &= 0, & \text{on } \partial\Omega, \\ \kappa \nabla \psi \cdot \mathbf{n} &= 0, & \text{on } \partial\Omega. \end{aligned}$$

Assume the above dual problem has the following regularity estimate

$$(7.2) \quad \|\psi\|_4 \leq C \|e_0\|.$$

THEOREM 7.1. Let  $u_h \in V_h$  be the solution of the weak Galerkin algorithm (3.1) with finite elements of order  $k \geq 2$ . Let  $t_0 = \min\{k, 3\}$ . Assume that the exact solution of (1.1) is sufficiently regular such that  $u \in H^{\max\{k+1, 4\}}(\Omega)$ , and the dual problem (7.1) has the  $H^4$  regularity. Then, there exists a constant  $C$  such that

$$(7.3) \quad \|Q_0 u - u_0\| \leq C h^{k+t_0-2} \left( \|u\|_{k+1} + h \delta_{k,2} \|u\|_4 \right).$$

In other words, we have a sub-optimal order of convergence for  $k = 2$  and optimal order of convergence for  $k \geq 3$ .

*Proof.* By testing the first equation of (7.1) against the error function  $e_0$  on each element and using the integration by parts, we obtain

$$\begin{aligned} \|e_0\|^2 &= (F^2 \psi, e_0) \\ &= \sum_{T \in \mathcal{T}_h} \left\{ (E\psi, Ee_0)_T - \langle E\psi, \kappa \nabla e_0 \cdot \mathbf{n} \rangle_{\partial T} + \langle \kappa \nabla(E\psi) \cdot \mathbf{n}, e_0 \rangle_{\partial T} \right. \\ &\quad \left. + (-2\mu E\psi + \mu^2 \psi, e_0)_T \right\} \\ &= \sum_{T \in \mathcal{T}_h} \left\{ (E\psi, Ee_0)_T - \langle E\psi, \kappa \nabla e_0 \cdot \mathbf{n} - e_g \rangle_{\partial T} + \langle \kappa \nabla(E\psi) \cdot \mathbf{n}, e_0 - e_b \rangle_{\partial T} \right. \\ &\quad \left. + (-2\mu E\psi + \mu^2 \psi, e_0)_T \right\}, \end{aligned}$$

where the added terms associated with  $e_b$  and  $e_g$  vanish due to the cancelation for interior edges and the fact that  $e_b$  and  $e_g$  have zero value on  $\partial\Omega$ . Using (5.4) with  $\psi$  and  $e_h$  in the place of  $u$  and  $v$  respectively, we arrive at

$$\begin{aligned} \|e_0\|^2 &= \sum_{T \in \mathcal{T}_h} \left\{ (E_w(Q_h \psi), E_w e_h)_T - \langle e_0 - e_b, \kappa \nabla(Q_h(E\psi)) \cdot \mathbf{n} \rangle_{\partial T} \right. \\ &\quad + \langle \kappa \nabla e_0 \cdot \mathbf{n} - e_g, Q_h E\psi \rangle_{\partial T} \\ &\quad - \langle E\psi, \kappa \nabla e_0 \cdot \mathbf{n} - e_g \rangle_{\partial T} + \langle \kappa \nabla(E\psi) \cdot \mathbf{n}, e_0 - e_b \rangle_{\partial T} \\ &\quad \left. + (-2\mu E\psi + \mu^2 \psi, e_0)_T \right\} \\ (7.4) \quad &= \sum_{T \in \mathcal{T}_h} \left\{ (E_w(Q_h \psi), E_w e_h)_T - \langle e_0 - e_b, \kappa \nabla(Q_h(E\psi) - E\psi) \cdot \mathbf{n} \rangle_{\partial T} \right. \\ &\quad \left. + \langle \kappa \nabla e_0 \cdot \mathbf{n} - e_g, Q_h E\psi - E\psi \rangle_{\partial T} + (-2\mu E\psi + \mu^2 \psi, e_0)_T \right\}. \end{aligned}$$

Next, it follows from the error equation (5.2) with  $v = Q_h \psi$  that

$$\begin{aligned} (7.5) \quad &(E_w e_h, E_w Q_h \psi)_h \\ &= -2\mu(\kappa \nabla_w e_h, \nabla_w Q_h \psi)_h - \mu^2(e_h, Q_h \psi)_h - s(e_h, Q_h \psi) + \phi_u(Q_h \psi). \end{aligned}$$

Substituting (7.5) into (7.4), combining by (2.5) with  $\psi = \nabla_w Q_h \psi$ , and from

(4.4), we obtain

$$\begin{aligned}
& \|e_0\|^2 \\
&= -2\mu(\kappa\nabla_w e_h, \nabla_w Q_h \psi)_h - \mu^2(e_h, Q_h \psi)_h - s(e_h, Q_h \psi) \\
&\quad - \sum_{T \in \mathcal{T}_h} \left\{ \langle \kappa \nabla(Eu - \mathcal{Q}_h(Eu)) \cdot \mathbf{n}, Q_0 \psi - Q_b \psi \rangle_{\partial T} \right. \\
&\quad + \langle \kappa \nabla Q_0 \psi \cdot \mathbf{n} - Q_g(\kappa \nabla \psi \cdot \mathbf{n}), Eu - \mathcal{Q}_h Eu \rangle_{\partial T} \\
&\quad + 2\mu \langle Q_0 \psi - Q_b \psi, (\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n} \rangle_{\partial T} \\
&\quad - \langle e_0 - e_b, \kappa \nabla(\mathcal{Q}_h(E\psi) - E\psi) \cdot \mathbf{n} \rangle_{\partial T} \\
&\quad \left. + \langle \kappa \nabla e_0 \cdot \mathbf{n} - e_g, \mathcal{Q}_h E\psi - E\psi \rangle_{\partial T} + (-2\mu E\psi + \mu^2 \psi, e_0)_T \right\} + s(Q_h u, Q_h \psi) \\
(7.6) \quad &= \sum_{T \in \mathcal{T}_h} \left\{ (\mu^2(\psi - Q_0 \psi) - 2\mu \nabla \cdot (\kappa \nabla \psi - \mathcal{Q}_1(\kappa \nabla \psi)), e_0)_T \right. \\
&\quad - 2\mu \kappa \langle e_b, \mathcal{Q}_1(\nabla \psi) \cdot \mathbf{n} \rangle_{\partial T} - \langle \kappa \nabla(Eu - \mathcal{Q}_h(Eu)) \cdot \mathbf{n}, Q_0 \psi - Q_b \psi \rangle_{\partial T} \\
&\quad + \langle \kappa \nabla Q_0 \psi \cdot \mathbf{n} - Q_g(\kappa \nabla \psi \cdot \mathbf{n}), Eu - \mathcal{Q}_h Eu \rangle_{\partial T} \\
&\quad + 2\mu \langle Q_0 \psi - Q_b \psi, (\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n} \rangle_{\partial T} \\
&\quad - \langle e_0 - e_b, \kappa \nabla(\mathcal{Q}_h(E\psi) - E\psi) \cdot \mathbf{n} \rangle_{\partial T} \\
&\quad \left. + \langle \kappa \nabla e_0 \cdot \mathbf{n} - e_g, \mathcal{Q}_h E\psi - E\psi \rangle_{\partial T} \right\} - s(e_h, Q_h \psi) + s(Q_h u, Q_h \psi) \\
&= \sum_{T \in \mathcal{T}_h} \left\{ (\mu^2(\psi - Q_0 \psi) - 2\mu \nabla \cdot (\kappa \nabla \psi - \mathcal{Q}_1(\kappa \nabla \psi)), e_0)_T \right. \\
&\quad - \langle \kappa \nabla(Eu - \mathcal{Q}_h(Eu)) \cdot \mathbf{n}, Q_0 \psi - Q_b \psi \rangle_{\partial T} \\
&\quad + \langle \kappa \nabla Q_0 \psi \cdot \mathbf{n} - Q_g(\kappa \nabla \psi \cdot \mathbf{n}), Eu - \mathcal{Q}_h Eu \rangle_{\partial T} \\
&\quad + 2\mu \langle Q_0 \psi - Q_b \psi, (\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n} \rangle_{\partial T} \\
&\quad - \langle e_0 - e_b, \kappa \nabla(\mathcal{Q}_h(E\psi) - E\psi) \cdot \mathbf{n} \rangle_{\partial T} \\
&\quad \left. + \langle \kappa \nabla e_0 \cdot \mathbf{n} - e_g, \mathcal{Q}_h E\psi - E\psi \rangle_{\partial T} \right\} - s(e_h, Q_h \psi) + s(Q_h u, Q_h \psi),
\end{aligned}$$

where we use the third equation of (7.1) to obtain

$$\sum_{T \in \mathcal{T}_h} 2\mu \kappa \langle e_b, \mathcal{Q}_1(\nabla \psi) \cdot \mathbf{n} \rangle_{\partial T} = 2\mu \kappa \langle e_b, \mathcal{Q}_1(\nabla \psi) \cdot \mathbf{n} \rangle_{\partial \Omega} = 2\mu \langle e_b, \kappa \nabla \psi \cdot \mathbf{n} \rangle_{\partial \Omega} = 0.$$

Each of these terms on the right-hand side of (7.6) can be bounded as follows. Note that  $t_0 = \min\{3, k\} \leq 3$ . For the first term of the right-hand side of (7.6), from

the Cauchy-Schwarz inequality, (7.2), the estimates (4.5) and (4.7), we have

$$\begin{aligned}
& \left| \sum_{T \in \mathcal{T}_h} (\mu^2(\psi - Q_0\psi) - 2\mu \nabla \cdot (\kappa \nabla \psi - \mathcal{Q}_1(\kappa \nabla \psi)), e_0)_T \right| \\
& \leq \left( \sum_{T \in \mathcal{T}_h} \|\psi - Q_0\psi\|_T^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} \|e_0\|_T^2 \right)^{\frac{1}{2}} \\
& \quad + \left( \sum_{T \in \mathcal{T}_h} \|\nabla \cdot (\kappa \nabla \psi - \mathcal{Q}_1(\kappa \nabla \psi))\|_T^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} \|e_0\|_T^2 \right)^{\frac{1}{2}} \\
& \leq Ch^{k+1} \|\psi\|_{k+1} \|e_0\| + Ch^{k-1} \|\psi\|_{k+1} \|e_0\| \\
& \leq Ch^{k-1} \|\psi\|_{k+1} \|e_0\| \\
& \leq Ch \|\psi\|_3 \|e_0\| \\
& \leq Ch \|\psi\|_4 \|e_0\| \\
& \leq Ch \|e_0\|^2.
\end{aligned}$$

For the second term of the right-hand side of (7.6), it follows from the Cauchy-Schwarz inequality and the estimates (4.9) and (4.11) that

$$\begin{aligned}
& \left| \sum_{T \in \mathcal{T}_h} 2\mu \langle \kappa \nabla (Eu - \mathcal{Q}_h Eu) \cdot \mathbf{n}, Q_0\psi - Q_b\psi \rangle_{\partial T} \right| \\
& \leq \left( \sum_{T \in \mathcal{T}_h} h_T^3 \|\kappa \nabla (Eu - \mathcal{Q}_h Eu) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_0\psi - Q_b\psi\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \leq Ch^{k-1} (\|u\|_{k+1} + h\delta_{k,2} \|u\|_4) h^{t_0-1} \|\psi\|_{t_0+1} \\
& \leq Ch^{k+t_0-2} (\|u\|_{k+1} + h\delta_{k,2} \|u\|_4) \|\psi\|_4.
\end{aligned}$$

For the third term of the right-hand side of (7.6), it follows from the Cauchy-Schwarz inequality and the estimates (4.8) and (4.10) that

$$\begin{aligned}
& \left| \sum_{T \in \mathcal{T}_h} \langle \kappa \nabla Q_0\psi \cdot \mathbf{n} - Q_g(\kappa \nabla \psi \cdot \mathbf{n}), Eu - \mathcal{Q}_h Eu \rangle_{\partial T} \right| \\
& \leq \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\kappa \nabla Q_0\psi \cdot \mathbf{n} - Q_g(\kappa \nabla \psi \cdot \mathbf{n})\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T \|Eu - \mathcal{Q}_h Eu\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \leq Ch^{t_0-1} \|\psi\|_{t_0+1} h^{k-1} \|u\|_{k+1} \\
& \leq Ch^{k+t_0-2} \|u\|_{k+1} \|\psi\|_4.
\end{aligned}$$

For the fourth term of the right-hand side of (7.6), it follows from the Cauchy-Schwarz inequality and the estimates (4.11) and (4.12) that

$$\begin{aligned}
& \left| 2\mu \langle Q_0\psi - Q_b\psi, (\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n} \rangle_{\partial T} \right| \\
& \leq \left( \sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_0\psi - Q_b\psi\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^3 \|(\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \leq Ch^{t_0-1} \|\psi\|_{t_0+1} h^{k+1} \|u\|_{k+1} \\
& \leq Ch^{k+t_0} \|u\|_{k+1} \|\psi\|_4.
\end{aligned}$$



For the fifth term of the right-hand side of (7.6), it follows from the Cauchy-Schwarz inequality and the estimate (4.9) that

$$\begin{aligned}
& \left| \sum_{T \in \mathcal{T}_h} \langle \kappa \nabla (E\psi - \mathcal{Q}_h E\psi) \cdot \mathbf{n}, e_0 - e_b \rangle_{\partial T} \right| \\
& \leq \left( \sum_{T \in \mathcal{T}_h} h_T^3 \|\kappa \nabla (E\psi - \mathcal{Q}_h E\psi) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^{-3} \|e_0 - e_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \leq Ch^{t_0-1} (\|\psi\|_{t_0+1} + h\delta_{t_0,2} \|\psi\|_4) \|e_h\| \\
& \leq Ch^{t_0-1} \|\psi\|_4 \|e_h\|.
\end{aligned}$$

For the sixth term of the right-hand side of (7.6), it follows from the Cauchy-Schwarz inequality and the estimate (4.8) that

$$\begin{aligned}
& \left| \sum_{T \in \mathcal{T}_h} \langle \kappa \nabla e_0 \cdot \mathbf{n} - e_g, E\psi - \mathcal{Q}_h E\psi \rangle_{\partial T} \right| \\
& \leq \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\kappa \nabla e_0 \cdot \mathbf{n} - e_g\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T \|E\psi - \mathcal{Q}_h E\psi\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \leq Ch^{t_0-1} \|\psi\|_{t_0+1} \|e_h\| \\
& \leq Ch^{t_0-1} \|\psi\|_4 \|e_h\|.
\end{aligned}$$

For the seventh term of the right-hand side of (7.6), it follows from the Cauchy-Schwarz inequality and the estimates (4.10) and (4.11) that

$$\begin{aligned}
& \left| s(Q_h \psi, e_h) \right| \\
& \leq \left| \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle \kappa \nabla Q_0 \psi \cdot \mathbf{n} - Q_g(\kappa \nabla \psi \cdot \mathbf{n}), \kappa \nabla e_0 \cdot \mathbf{n} - e_g \rangle \right| \\
& \quad + \left| \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle Q_0 \psi - Q_b \psi, e_0 - e_b \rangle \right| \\
& \leq \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\kappa \nabla Q_0 \psi \cdot \mathbf{n} - Q_g(\kappa \nabla \psi \cdot \mathbf{n})\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\kappa \nabla e_0 \cdot \mathbf{n} - e_g\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \quad + \left( \sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_0 \psi - Q_b \psi\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^{-3} \|e_0 - e_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \leq Ch^{t_0-1} \|\psi\|_{t_0+1} \|e_h\| \\
& \leq Ch^{t_0-1} \|\psi\|_4 \|e_h\|.
\end{aligned}$$

For the last term of the right-hand side of (7.6), it follows from the Cauchy-Schwarz

inequality and the estimates (4.10) and (4.11) that

$$\begin{aligned}
& \left| s(Q_h u, Q_h \psi) \right| \\
& \leq \left| \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle \kappa \nabla Q_0 u \cdot \mathbf{n} - Q_g(\kappa \nabla u \cdot \mathbf{n}), \kappa \nabla Q_0 \psi \cdot \mathbf{n} - Q_g(\kappa \nabla \psi \cdot \mathbf{n}) \rangle_{\partial T} \right| \\
& \quad + \left| \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle Q_0 u - Q_b u, Q_0 \psi - Q_b \psi \rangle_{\partial T} \right| \\
& \leq \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\kappa \nabla Q_0 u \cdot \mathbf{n} - Q_g(\kappa \nabla u \cdot \mathbf{n})\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \quad \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\kappa \nabla Q_0 \psi \cdot \mathbf{n} - Q_g(\kappa \nabla \psi \cdot \mathbf{n})\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \quad + \left( \sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_0 u - Q_b u\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_0 \psi - Q_b \psi\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \leq C h^{t_0-1} \|\psi\|_{t_0+1} h^{k-1} \|u\|_{k+1} \\
& \leq C h^{k+t_0-2} \|u\|_{k+1} \|\psi\|_4.
\end{aligned}$$

Finally, by substituting all the above estimates into (7.6), we obtain

$$(1 - Ch) \|e_0\|^2 \leq C(h^{t_0-1} \|e_h\| + h^{k+t_0-2} (\|u\|_{k+1} + h\delta_{k,2} \|u\|_4)) \|\psi\|_4,$$

which, together with the regularity estimate (7.2) and (6.1), gives rise to the desired  $L^2$  error estimate (7.3). This completes the proof of the theorem.  $\square$

**THEOREM 7.2.** *Let  $u_h \in V_h$  be the solution of the weak Galerkin algorithm (3.1) with finite elements of order  $k \geq 2$ . Let  $t_0 = \min\{k, 3\}$ . Assume that the exact solution of (1.1) is sufficiently regular such that  $u \in H^{\max\{k+1, 4\}}(\Omega)$ , and the dual problem (7.1) has the  $H^4$  regularity. Define*

$$\|u_b\| = \left( \sum_{T \in \mathcal{T}_h} h_T \|u_b\|_{\partial T}^2 \right)^{\frac{1}{2}}.$$

*There exists a constant  $C$  such that*

$$\|Q_b u - u_b\| \leq C h^{k+t_0-2} (\|u\|_{k+1} + h\delta_{k,2} \|u\|_4).$$

*In other words, we have a sub-optimal order of convergence for  $k = 2$  and optimal order of convergence for  $k \geq 3$ .*

*Proof.* Letting  $v = \{0, e_b, 0\}$  in the error equation (5.2), we obtain

$$\begin{aligned}
& (E_w e_h, E_w v)_h + 2\mu(\kappa \nabla_w e_h, \nabla_w v)_h + \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle e_0 - e_b, -e_b \rangle_{\partial T} \\
(7.7) \quad & = - \sum_{T \in \mathcal{T}_h} \langle \kappa \nabla(Eu - \mathcal{Q}_h(Eu)) \cdot \mathbf{n}, -e_b \rangle_{\partial T} \\
& \quad + \sum_{T \in \mathcal{T}_h} 2\mu \langle -e_b, (\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n} \rangle_{\partial T} + \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle Q_0 u - Q_b u, -e_b \rangle_{\partial T}.
\end{aligned}$$

Letting  $\varphi = E_w e_h$  in (2.2) and  $\psi = \nabla_w e_h$  in (2.5) yields

$$(E_w v, E_w e_h)_T = -\langle e_b, \kappa \nabla(E_w e_h) \cdot \mathbf{n} \rangle_{\partial T},$$

$$(\nabla_w v, \nabla_w e_h)_T = \langle e_b, \nabla_w e_h \cdot \mathbf{n} \rangle_{\partial T},$$

which, combining with re-arranging the terms involved in (7.7), give

$$(7.8) \quad \begin{aligned} & \sum_{T \in \mathcal{T}_h} h_T^{-3} \|e_b\|_{\partial T}^2 \\ &= \sum_{T \in \mathcal{T}_h} \{ \langle e_b, \kappa \nabla(E_w e_h) \cdot \mathbf{n} \rangle_{\partial T} - 2\mu \kappa \langle e_b, \nabla_w e_h \cdot \mathbf{n} \rangle_{\partial T} + h_T^{-3} \langle e_0, e_b \rangle_{\partial T} \\ & \quad + \langle \kappa \nabla(Eu - \mathcal{Q}_h(Eu)) \cdot \mathbf{n}, e_b \rangle_{\partial T} \\ & \quad + 2\mu \langle -e_b, (\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n} \rangle_{\partial T} + h_T^{-3} \langle \mathcal{Q}_0 u - \mathcal{Q}_b u, -e_b \rangle_{\partial T} \}. \end{aligned}$$

Each term on the right-hand side of (7.8) can be bounded as follows. For the first term of the right-hand side of (7.8), using Cauchy-Schwarz inequality, the trace inequality (4.2) and inverse inequality, we obtain

$$\begin{aligned} & \left| \sum_{T \in \mathcal{T}_h} \langle e_b, \kappa \nabla(E_w e_h) \cdot \mathbf{n} \rangle_{\partial T} \right| \\ & \leq \left( \sum_{T \in \mathcal{T}_h} h_T \|e_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\kappa \nabla(E_w e_h) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ & \leq C \left( \sum_{T \in \mathcal{T}_h} h_T \|e_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^{-4} \|E_w e_h\|_T^2 \right)^{\frac{1}{2}} \\ & \leq C h^{-2} \|e_b\| \|e_h\|. \end{aligned}$$

For the second term of the right-hand side of (7.8), using Cauchy-Schwarz inequality and the trace inequality (4.2), we obtain

$$\begin{aligned} & \left| \sum_{T \in \mathcal{T}_h} 2\mu \kappa \langle e_b, \nabla_w e_h \cdot \mathbf{n} \rangle_{\partial T} \right| \\ & \leq \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\nabla_w e_h \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T \|e_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ & \leq C h^{-1} \|e_b\| \|e_h\|. \end{aligned}$$

For the third term of the right-hand side of (7.8), using Cauchy-Schwarz inequality and the trace inequality (4.2), we obtain

$$\begin{aligned} & \left| \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle e_0, e_b \rangle_{\partial T} \right| \\ & \leq \left( \sum_{T \in \mathcal{T}_h} h_T^{-7} \|e_0\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T \|e_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ & \leq C h^{-4} \|e_b\| \|e_0\|. \end{aligned}$$

For the fourth term of the right-hand side of (7.8), using Cauchy-Schwarz inequality and the estimate (4.9), we obtain

$$\begin{aligned}
& \left| \sum_{T \in \mathcal{T}_h} \langle \kappa \nabla (Eu - \mathcal{Q}_h Eu) \cdot \mathbf{n}, -e_b \rangle_{\partial T} \right| \\
& \leq Ch^{-2} \left( \sum_{T \in \mathcal{T}_h} h_T^3 \|\kappa \nabla (Eu - \mathcal{Q}_h Eu) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T \|e_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \leq Ch^{k-3} \|e_b\| (\|u\|_{k+1} + h\delta_{k,2} \|u\|_4).
\end{aligned}$$

For the fifth term of the right-hand side of (7.8), using Cauchy-Schwarz inequality and the estimate (4.12), we obtain

$$\begin{aligned}
& \left| \sum_{T \in \mathcal{T}_h} 2\mu \langle -e_b, (\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n} \rangle_{\partial T} \right| \\
& \leq Ch^{-2} \left( \sum_{T \in \mathcal{T}_h} h_T \|e_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^3 \|(\kappa \nabla u - \mathcal{Q}_1(\kappa \nabla u)) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \leq Ch^{k-1} \|e_b\| \|u\|_{k+1}.
\end{aligned}$$

For the last term of the right-hand side of (7.8), using Cauchy-Schwarz inequality and the estimate (4.11), we obtain

$$\begin{aligned}
& \left| \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle Q_0 u - Q_b u, -e_b \rangle_{\partial T} \right| \\
& \leq Ch^{-2} \left( \sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_0 u - Q_b u\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T \|e_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \leq Ch^{k-3} \|e_b\| \|u\|_{k+1}.
\end{aligned}$$

Substituting all the above estimates into (7.8) and the shape-regularity assumption for the finite element partition  $\mathcal{T}_h$  give

$$\begin{aligned}
h^{-4} \|e_b\|^2 & \leq C \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle e_b, e_b \rangle_{\partial T} \\
& \leq C \left( h^{-2} \|e_h\| + h^{-1} \|e_h\| + h^{-4} \|e_0\| + h^{k-3} (\|u\|_{k+1} \right. \\
& \quad \left. + h\delta_{k,2} \|u\|_4) + h^{k-1} \|u\|_{k+1} + h^{k-3} \|u\|_{k+1} \right) \|e_b\|,
\end{aligned}$$

which, combining with (7.3) and (6.1), yields

$$\begin{aligned}
\|e_b\| & \leq C \left( h^2 \|e_h\| + \|e_0\| + h^{k+1} (\|u\|_{k+1} + h\delta_{k,2} \|u\|_4) + h^{k+1} \|u\|_{k+1} \right) \\
& \leq C \left( h^2 h^{k-1} (\|u\|_{k+1} + h\delta_{k,2} \|u\|_4) + h^{k+t_0-2} (\|u\|_{k+1} + h\delta_{k,2} \|u\|_4) \right. \\
& \quad \left. + h^{k+1} (\|u\|_{k+1} + h\delta_{k,2} \|u\|_4) + h^{k+1} \|u\|_{k+1} \right) \\
& \leq Ch^{k+t_0-2} (\|u\|_{k+1} + h\delta_{k,2} \|u\|_4).
\end{aligned}$$

This completes the proof.  $\square$

**THEOREM 7.3.** *Let  $u_h \in V_h$  be the solution of the weak Galerkin algorithm (3.1) with finite elements of order  $k \geq 2$ . Let  $t_0 = \min\{k, 3\}$ . Assume that the exact*

solution of (1.1) is sufficiently regular such that  $u \in H^{\max\{k+1,4\}}(\Omega)$ , and the dual problem (7.1) has the  $H^4$  regularity. Define

$$\|u_g\| = \left( \sum_{T \in \mathcal{T}_h} h_T \|u_g\|_{\partial T}^2 \right)^{\frac{1}{2}}.$$

There exists a constant  $C$  such that

$$\|Q_g(\kappa \nabla u_0 \cdot \mathbf{n}) - u_g\| \leq Ch^{k+t_0-3} \left( \|u\|_{k+1} + h\delta_{k,2} \|u\|_4 \right).$$

In other words, we have a sub-optimal order of convergence for  $k = 2$  and optimal order of convergence for  $k \geq 3$ .

*Proof.* Letting  $v = \{0, 0, e_g\}$  in the error equation (5.2), we obtain

$$\begin{aligned} (7.9) \quad & (E_w e_h, E_w v)_h + 2\mu(\kappa \nabla_w e_h, \nabla_w v)_h + \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle \kappa \nabla e_0 \cdot \mathbf{n} - e_g, -e_g \rangle_{\partial T} \\ & = \sum_{T \in \mathcal{T}_h} \langle -e_g, Eu - \mathcal{Q}_h Eu \rangle_{\partial T} + \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle \kappa \nabla Q_0 u \cdot \mathbf{n} - Q_g(\kappa \nabla u \cdot \mathbf{n}), -e_g \rangle_{\partial T}. \end{aligned}$$

Letting  $\varphi = E_w e_h$  in (2.2) and  $\psi = \nabla_w e_h$  in (2.5) yields

$$(E_w v, E_w e_h)_T = \langle e_g, E_w e_h \rangle_{\partial T},$$

$$(\nabla_w v, \nabla_w e_h)_T = 0,$$

which, combining with re-arranging the terms involved in (7.9), yield

$$\begin{aligned} (7.10) \quad & \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle e_g, e_g \rangle_{\partial T} = \sum_{T \in \mathcal{T}_h} -\langle e_g, E_w e_h \rangle_{\partial T} + \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle \kappa \nabla e_0 \cdot \mathbf{n}, e_g \rangle_{\partial T} \\ & - \sum_{T \in \mathcal{T}_h} \langle e_g, Eu - \mathcal{Q}_h Eu \rangle_{\partial T} + \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle \kappa \nabla Q_0 u \cdot \mathbf{n} - Q_g(\kappa \nabla u_0 \cdot \mathbf{n}), -e_g \rangle_{\partial T}. \end{aligned}$$

Each term on the right-hand side of (7.10) can be bounded as follows. For the first term of the right-hand side of (7.10), using Cauchy-Schwarz inequality, the trace inequality (4.2), we obtain

$$\begin{aligned} & \left| \sum_{T \in \mathcal{T}_h} \langle e_g, E_w e_h \rangle_{\partial T} \right| \\ & \leq \left( \sum_{T \in \mathcal{T}_h} h_T \|e_g\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|E_w e_h\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ & \leq C \left( \sum_{T \in \mathcal{T}_h} h_T \|e_g\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^{-2} \|E_w e_h\|_T^2 \right)^{\frac{1}{2}} \\ & \leq Ch^{-1} \|e_g\| \|e_h\|. \end{aligned}$$

For the second term of the right-hand side of (7.10), using Cauchy-Schwarz inequality

and the trace inequality (4.2) and inverse inequality, we obtain

$$\begin{aligned}
& \left| \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle \kappa \nabla e_0 \cdot \mathbf{n}, e_g \rangle_{\partial T} \right| \\
& \leq \left( \sum_{T \in \mathcal{T}_h} h_T^{-3} \|\kappa \nabla e_0 \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T \|e_g\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \leq \left( \sum_{T \in \mathcal{T}_h} h_T^{-6} \|e_0\|_T^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T \|e_g\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \leq C h^{-3} \|e_g\| \|e_0\|.
\end{aligned}$$

For the third term of the right-hand side of (7.10), using Cauchy-Schwarz inequality and the estimate (4.8), we obtain

$$\begin{aligned}
& \left| \sum_{T \in \mathcal{T}_h} \langle e_g, Eu - \mathcal{Q}_h Eu \rangle_{\partial T} \right| \\
& \leq C h^{-1} \left( \sum_{T \in \mathcal{T}_h} h_T \|e_g\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T \|Eu - \mathcal{Q}_h Eu\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \leq C h^{k-2} \|e_g\| \|u\|_{k+1}.
\end{aligned}$$

For the last term of the right-hand side of (7.10), using Cauchy-Schwarz inequality and the estimate (4.10), we obtain

$$\begin{aligned}
& \left| \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle \kappa \nabla Q_0 u \cdot \mathbf{n} - Q_g(\kappa \nabla u_0 \cdot \mathbf{n}), -e_g \rangle_{\partial T} \right| \\
& \leq C h^{-1} \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\kappa \nabla Q_0 u \cdot \mathbf{n} - Q_g(\kappa \nabla u_0 \cdot \mathbf{n})\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T \|e_g\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \leq C h^{k-2} \|e_g\| \|u\|_{k+1}.
\end{aligned}$$

Substituting all the above estimates into (7.10) and the shape-regularity assumption for the finite element partition  $\mathcal{T}_h$  give

$$\begin{aligned}
h^{-2} \|e_g\|^2 & \leq C \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle e_g, e_g \rangle_{\partial T} \\
& \leq C (h^{-1} \|e_h\| + h^{-3} \|e_0\| + h^{k-2} \|u\|_{k+1}) \|e_g\|,
\end{aligned}$$

which, together with (7.3) and (6.1), gives

$$\begin{aligned}
\|e_g\| & \leq C (h \|e_h\| + h^{-1} \|e_0\| + h^k \|u\|_{k+1}) \\
& \leq C \left( h \cdot h^{k-1} (\|u\|_{k+1} + h \delta_{k,2} \|u\|_4) + h^{-1} \cdot h^{k+t_0-2} (\|u\|_{k+1} + h \delta_{k,2} \|u\|_4) + h^k \|u\|_{k+1} \right) \\
& \leq C h^{k+t_0-3} (\|u\|_{k+1} + h \delta_{k,2} \|u\|_4).
\end{aligned}$$

This completes the proof.  $\square$

**8. Numerical Tests.** In this section, we present some numerical results for the WG finite element method analyzed in the previous sections. The goal is to

demonstrate the efficiency and the convergence theory established for the method. For the simplicity of implementation, the weak function  $v = \{v_0, v_b, v_g\}$  can be discretized by polynomials of degree of  $k$ ,  $k - 1$  and  $k - 1$ , respectively. We could obtain all the same error estimates as obtained in the previous sections and the analysis could be derived without any difficulty[32]. Details are omitted here.

In our experiments, we implement the lowest order (i.e.,  $k = 2$ ) scheme for the weak Galerkin algorithm (3.1). In other words, the implementation makes use of the following finite element space

$$\tilde{V}_h = \{v = \{v_0, v_b, v_g\}, v_0 \in P_2(T), v_b \in P_1(e), v_g \in P_1(e), T \in \mathcal{T}_h, e \in \mathcal{E}_h\}.$$

For any given  $v = \{v_0, v_b, v_g\} \in \tilde{V}_h$ , the discrete weak second order elliptic operator  $E_w v$  is computed as a constant locally on each element  $T$  by solving the following equation

$$(E_w v, \varphi)_T = (v_0, E\varphi)_T - \langle v_b, \kappa \nabla \varphi \cdot \mathbf{n} \rangle_{\partial T} + \langle v_g, \varphi \rangle_{\partial T},$$

for all  $\varphi \in P_0(T)$ . Since  $\varphi \in P_0(T)$ , the above equation can be simplified as

$$(E_w v, \varphi)_T = \langle v_g, \varphi \rangle_{\partial T}.$$

The WG finite element scheme (3.1) was implemented on uniform triangular partition, which was obtained by partitioning the domain into  $n \times n$  sub-squares and then dividing each square element into two triangles by the diagonal line with a negative slope. The mesh size is denoted by  $h = 1/n$ .

Table 8.1 shows the numerical results for the exact solution  $u = x^2(1-x)^2y^2(1-y)^2$ . The numerical experiment is conducted on the unit square domain  $\Omega = (0, 1)^2$ . This case has homogeneous boundary conditions for both Dirichlet and Neumann. We take the coefficient matrix  $\kappa = [1/(3(1+0.01)), 0; 0, 1/(3(1+0.01))]$  and  $\mu = 0.01$  in the whole domain  $\Omega$ . The results indicate that the convergence rate for the solution of the weak Galerkin algorithm (3.1) is of order  $O(h)$  in the discrete  $H^2$  norm, and is of order  $O(h^2)$  in the standard  $L^2$  norm. The numerical results are in good consistency with theory for the  $H^2$  and  $L^2$  norm of the error. Figure 8.1 illustrates the WG numerical solution for the mesh size  $1/64$ , which totals to 4096 elements.

TABLE 8.1  
Numerical error and convergence order for the exact solution  $u = x^2(1-x)^2y^2(1-y)^2$ .

$1/n$	$\ u_0 - Q_0 u\ $	order in $L^2$ norm	$\ u_h - Q_h u\ $	order in $H^2$ norm
1	0.05458		0.09913	
2	0.02163	1.33	0.06649	0.58
4	0.006307	1.78	0.03643	0.87
8	0.001716	1.88	0.01904	0.94
16	4.582e-04	1.91	0.009830	0.95
32	1.181e-04	1.96	0.004988	0.98
64	2.981e-05	1.99	0.002506	0.99

Table 8.2 presents the numerical results when the exact solution is given by  $u = \sin(\pi x)\sin(\pi y)$  on the unit square domain  $\Omega = (0, 1)^2$ , which has nonhomogeneous

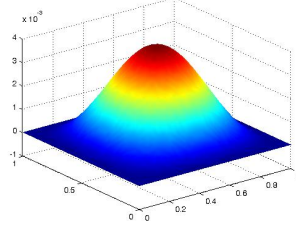


FIG. 8.1. WG finite element solution for the exact solution  $u = x^2(1-x)^2y^2(1-y)^2$  with mesh size  $1/64$ .

boundary conditions. The coefficient matrix  $\kappa$  and the constant  $\mu$  are taken to be the same values as the previous test. It shows that the convergence rates for the solution of the weak Galerkin algorithm (3.1) in the  $H^2$  and  $L^2$  norms are of order  $O(h)$  and  $O(h^2)$ , respectively, which are in consistency with theory for the  $L^2$  and  $H^2$  norms of the error. Figure 8.2 gives the WG numerical solution for the mesh size  $1/64$ .

TABLE 8.2  
Numerical error and convergence order for the exact solution  $u = \sin(\pi x)\sin(\pi y)$ .

$1/n$	$\ u_0 - Q_0 u\ $	order in $L^2$ norm	$\ u_h - Q_h u\ $	order in $H^2$ norm
1	5.587		10.20	
2	2.143	1.38	6.590	0.63
4	0.6017	1.83	3.526	0.90
8	0.1549	1.96	1.793	0.98
16	0.03904	1.99	0.9005	0.99
32	0.009783	2.00	0.4508	1.00
64	0.002447	2.00	0.2255	1.00

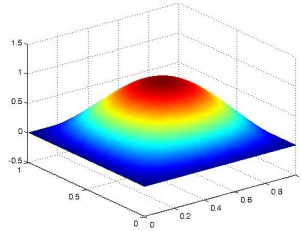


FIG. 8.2. WG finite element solution for the exact solution  $u = \sin(\pi x)\sin(\pi y)$  with mesh size  $1/64$ .

In the rest of this section, we shall conduct three different types of numerical tests arising from the FT model. Firstly, we take  $\kappa = [1, 0; 0, 1]$  in the subdomain  $\Omega_0 = (1/4, 3/8)^2$  and  $\kappa = [10^{-5}, 0; 0, 10^{-5}]$  in the rest of the whole domain  $\Omega = (0, 1)^2$ . We take  $\mu$  to be 0 in the whole domain  $\Omega$ . The right-hand side  $f$  is taken to be zero. For the Dirichlet boundary condition  $u = \xi$ , we take the boundary function  $\xi$  as a piecewise continuous function which takes value 1 on the middle of each boundary segment and takes value 0 on all the other edges for each of the four boundary edges.



As to the Neumann boundary condition  $\kappa \nabla u \cdot \mathbf{n} = \nu$ , we take  $\nu = -\xi$  in the test. Figure 8.3 illustrates the WG finite element solution for the mesh size  $1/64$ .

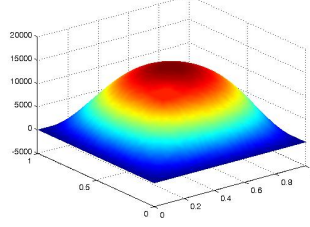


FIG. 8.3. WG finite element solution with discontinuous coefficients with mesh size  $1/64$ .

Secondly, we take the same  $\kappa$ ,  $\mu$  and  $f$  as in the previous test. The Dirichlet boundary condition  $u = \xi$  is set as an approximate Dirac- $\delta$  function on each of the four boundary edges. More precisely, this boundary data assumes value  $\frac{1}{|e|}$  on the middle edge of each boundary segment, and takes value 0 on all the other edges. As to the Neumann boundary condition  $\kappa \nabla u \cdot \mathbf{n} = \nu$ , we take  $\nu = -\xi$  in the test. Figure 8.4 illustrates the WG finite element solution for the mesh size  $1/64$ .

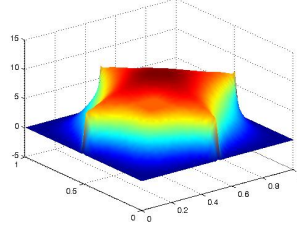


FIG. 8.4. WG finite element solution with discontinuous coefficients with mesh size  $1/64$  with different Dirichlet boundary condition set as an approximate Dirac- $\delta$  function.

At last, we consider a real problem arising from FT model which is implemented on the domain  $\Omega = (0, 50)^2$ . There are two blocks in the domain  $\Omega$ : one is at  $(25, 15)$  with radius 4; the other is at  $(35, 20)$  with radius 3. The right hand side data  $f$  is the function modeling the light source, and we use Gaussian function to model each point source  $(x_0, y_0)$ , with their centers locating around the boundary of the domain. More precisely, we set  $f = \sqrt{2\pi\epsilon} e^{-\frac{(x-x_0)^2 + (y-y_0)^2}{2\epsilon}}$  with  $\epsilon = 100/64$ . The coefficient matrix  $\kappa$  is taken to be  $[1/(3(1+0.01)), 0; 0, 1/(3(1+0.01))]$  and  $\mu$  is taken to be 0.01 in the whole domain  $\Omega$  which arise from the data of the real problem in FT. Figures 8.5-8.6 illustrate the WG finite element solution for light sources with the mesh size  $1/64$ , where the coordinates of light sources are  $(13.3065, 0.0730994)$  and  $(49.8272, 13.5234)$ , respectively.

In the future, we plan to conduct more numerical experiments for the weak Galerkin algorithm (3.1), particularly for elements of order higher than  $k = 2$  and for the real problems arising from the FT model. There is also a need of developing fast solution techniques for the matrix problem arising from the WG finite element scheme (3.1). Numerical experiments on finite element partitions with arbitrary polygonal element should be conducted for a further assessment of the WG method.

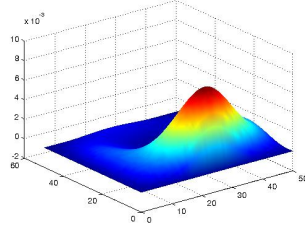


FIG. 8.5. WG finite element solution for a test case with source point  $(13.3065, 0.0730994)$  with mesh size  $1/64$ .

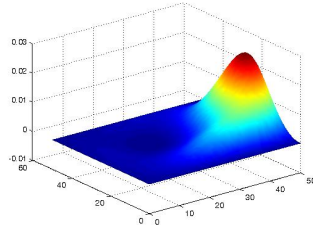


FIG. 8.6. WG finite element solution for a test case with source point  $(49.8272, 13.5234)$  with mesh size  $1/64$ .

## REFERENCES

- [1] D. ARNOLD AND F. BREZZI, *Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates*, RAIRO Modl. Math. Anal. Numr., vol. 19(1), pp.7-32, 1985.
- [2] S. BRENNER, V. NTZIACHRISTOS AND R. WEISSLEDER, *Optical-based molecular imaging: contrast agents and potential medical applications*, European Radiology, vol. 13, pp. 231-243, 2003.
- [3] S. BRENNER AND L. SCOTT, *The Mathematical Theory of Finite Element Methods, 3rd Edition*, 2008.
- [4] S. BRENNER AND L. SUNG,  *$C^0$  interior penalty methods for fourth order elliptic boundary value problems on polygonal domains*, J. Sci. Comput., pp. 83-118, 2005.
- [5] S. CHOW, K. YIN, H. ZHOU AND A. BEHROOZ, *Solving inverse source problems by the orthogonal solution and kernel correction algorithm(OSKCA) with applications in fluorescence tomography*, Inverse Problems and Imaging, vol. 8, pp. 79-102, 2014.
- [6] G. ENGEL, K. GARIKIPATI, T. HUGHES, M.G. LARSON, L. MAZZEI AND R. TAYLOR, *Continuous/discontinuous finite element approximations of fourth order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity*, Comput. Meth. Appl. Mech. Eng., vol. 191, pp. 3669-3750, 2002.
- [7] R. FALK, *Approximation of the biharmonic equation by a mixed finite element method*, SIAM J. Numer. Anal., vol. 15, pp. 556-567, 1978.
- [8] H. GAO AND H. ZHAO, *Analysis of a numerical solver for radiative transport equation*, Math. Comp., vol. 82, no. 281, pp. 153-172, 2013.
- [9] T. GUDI, N. NATARAJ AND A. K. PANI, *Mixed discontinuous Galerkin finite element method for the biharmonic equation*, J. Sci. Comput., vol. 37, pp. 139-161, 2008.
- [10] J. GUERMOND, G. KANSCHAT AND J. RAGUSA, *Discontinuous Galerkin for the radiative transport equation. Recent developments in discontinuous Galerkin finite element methods for partial differential equations*, IMA Vol. Math. Appl., 157, Springer, Cham, pp. 181-193, 2014.
- [11] D. HAWRYSZ AND E. SEVICK-MURACA, *Developments toward diagnostic breast cancer imaging using near-infrared optical measurements and fluorescent contrast agents*, Neoplasia (New

- York, NY), vol. 2, pp. 388-417, 2000.
- [12] J. HEBDEN, S. ARRIDGE AND D. DELPY, *Optical imaging in medicine: I. experimental techniques*, Physics in Medicine and Biology, vol. 42, pp. 825-840, 1997.
  - [13] O. LEHTIKANGAS, T. TARVAINEN, A. KIM AND S. ARRIDGE, *Finite element approximation of the radiative transport equation in a medium with piece-wise constant refractive index*, J. Comput. Phys., vol. 282, pp. 345-359, 2015.
  - [14] L. MU, Y. WANG, J. WANG AND X. YE, *A weak Galerkin mixed finite element method for biharmonic equations*, Numerical Solution of Partial Differential Equations: Theory, Algorithms, and Their Applications, vol. 45. pp. 247-277, 2013. arXiv:1210.3818v2.
  - [15] L. MU, J. WANG AND X. YE, *Weak Galerkin finite element methods for the biharmonic equation on polytopal meshes*, Numerical Methods for Partial Differential Equations, vol. 30, pp. 1003 -1029, 2014. arXiv:1303.0927v1.
  - [16] L. MU, J. WANG AND X. YE, *Weak Galerkin finite element methods on polytopal meshes*, International Journal of Numerical Analysis and Modeling, vol. 12, pp. 31-53, 2015. arXiv:1204.3655v2.
  - [17] S. MOHAN, T. TARVAINEN, M. SCHWEIGER, A. PULKKINEN AND S. ARRIDGE, *Variable order spherical harmonic expansion scheme for the radiative transport equation using finite elements*, J. Comput. Phys., vol. 230, no. 19, pp. 7364-7383, 2011.
  - [18] P. MONK, *A mixed finite element methods for the biharmonic equation*, SIAM J. Numer. Anal., vol. 24, pp. 737-749, 1987.
  - [19] L. MORLEY, *The triangular equilibrium element in the solution of plate bending problems*, Aero. Quart., vol. 19, pp. 149-169, 1968.
  - [20] I. MOZOLEVSKI, E. SLI AND P. BSING, : *hp-Version a priori error analysis of interior penalty discontinuous Galerkin finite element approximations to the biharmonic equation*, J. Sci. Comput., vol. 30, pp. 465-491, 2007.
  - [21] C. WANG AND J. WANG, *An Efficient Numerical Scheme for the Biharmonic Equation by Weak Galerkin Finite Element Methods on Polygonal or Polyhedral Meshes*, Comput. Math. Appl., vol. 68, no. 12, part B, pp. 2314-2330, 2014.
  - [22] C. WANG AND J. WANG, *A Hybridized Weak Galerkin Finite Element Method for the Biharmonic Equation*, Int. J. Numer. Anal. Model., vol. 12 , no. 2, pp. 302-317, 2015. arXiv:1402.1157.
  - [23] C. WANG AND J. WANG, *Discretization of div-curl Systems by Weak Galerkin Finite Element Methods on Polyhedral Partitions*, submitted to J. Sci. Comput., arXiv:1501.04616.
  - [24] C. WANG AND J. WANG, *A Locking-Free Weak Galerkin Finite Element Method for Elasticity Problems in the Primal Formulation* , submitted to Journal of Computers and Mathematics with Applications. arXiv:1508.05695.
  - [25] J. WANG AND X. YE, *A weak Galerkin finite element method for second-order elliptic problems*, J. Comp. and Appl. Math., vol. 241, pp. 103-115, 2013. arXiv:1104.2897v1.
  - [26] J. WANG AND X. YE, *A weak Galerkin mixed finite element method for second-order elliptic problems*. Math. Comp., vol. 83, no. 289, pp. 2101-2126, 2014. arXiv:1202.3655v2.
  - [27] J. WANG AND X. YE, *A weak Galerkin finite element method for the Stokes equations*. arXiv:1302.2707v1.
  - [28] J. WARSA, M. BENZI, T. WAREING AND J. MOREL, *Two-level preconditioning of a discontinuous Galerkin method for radiation diffusion*, Numerical mathematics and advanced applications, Springer Italia, Milan, pp. 967-977, 2003.
  - [29] J. WARSA, M. BENZI, T. WAREING AND J. MOREL, *Preconditioning a mixed discontinuous finite element method for radiation diffusion*, Numer. Linear Algebra Appl., vol. 11, no. 8-9, pp. 795-811, 2004.
  - [30] R. WEISSELEDER, C. TUNG, U. MAHMOOD AND A. BOGDANOV, *In vivo imaging of tumors with protease-activated near-infrared fluorescent probes*, Nature Biotechnology, vol. 17, pp. 375-378, 1999.
  - [31] K. YIN, *New Algorithms for solving inverse source problems in imaging techniques with applications in fluorescence tomography*, Ph.D. Thesis, Georgia Institute of Technology, 2013.
  - [32] R. ZHANG AND Q. ZHANG, *A weak Galerkin finite element scheme for the biharmonic equations by using polynomials of reduced order*, J. Sci. Comput., vol. 64, no. 2, pp. 559-585, 2015.